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TRANSVERSE VIBRATIONS OF SIMPLY SUPPORTED ANISOTROPIC RECTANGULAR PLATES CARRYING AN ELASTICALLY MOUNTED CONCENTRATED MASS

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1. Introduction

The use of structural elements of anisotropic characteristics is quite common in several fields of technology: from aerospace applications to the design of ocean structures, deep sumersibles, etc., passing through printed circuit boards (PCBs) used in electronic equipments [1]. In a great majority of circumstances these elements are used in dynamic environments and the design engineer confronts the challenge of finding their dynamic parameters: natural frequencies, mode shapes, etc. The present study deals with free vibrations of simply supported, rectangular, anisotropic plates.

The displacement amplitude is expressed in terms of a truncated double Fourier series and the four lower natural frequency coefficients are determined using the classical Rayleigh–Ritz method.

The present study undertakes two tasks: (1) it presents a series of numerical experiments whereby the convergence of the procedure is studied as the number of terms utilized is increased, when considering bare rectangular anisotropic plates. It is believed that the question of convergence is of basic mathematical and mechanics interest since the co-ordinate functions do not satisfy the natural conditions at the edges*[2]. (2) it analyzes the behavior of anisotropic plates carrying elastically mounted concentrated masses.

2. Approximate analytical solution

The Rayleigh–Ritz method requires minimization of the combined functional [3]

$$J[W', v'] = J_p[W'] + J_m[v'], \tag{1}$$

where $J_p[W']$ is the functional for the displacement amplitude of the plate and $J_m[v']$ is the corresponding functional for the displacement amplitude of the concentrated mass (see Figure 1). Each functional, in turn, has the general form

$$J = U - T, \tag{2}$$

U and T being the maximum strain energy and maximum kinetic energy of the plate and mass–spring system, respectively.

* The co-ordinate functions, instead, satisfy identically the geometric and the natural boundary conditions in the case of isotropic and orthotropic plates when the elastic axes are parallel to the boundaries of the plate.

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As is well known, see for example reference [2], in the case of rectangular plates of total general anisotropy its functional can be written as

$$J_p = \frac{1}{2}\int_{A_p}\left\{D_{11}\left(\frac{\partial^2 W'}{\partial x'^2}\right)^2 + 2D_{12}\frac{\partial^2 W'}{\partial x'^2}\frac{\partial^2 W'}{\partial y'^2} + D_{22}\left(\frac{\partial^2 W'}{\partial y'^2}\right)^2\right.$$

$$+ 4\left[D_{16}\left(\frac{\partial^2 W'}{\partial x'^2}\right) + D_{26}\left(\frac{\partial^2 W'}{\partial y'^2}\right)\right]\left(\frac{\partial^2 W'}{\partial x'\,\partial y'}\right)$$

$$\left. + 4D_{66}\left(\frac{\partial^2 W'}{\partial x'\,\partial y'}\right)^2\right\}\mathrm{d}x'\,\mathrm{d}y' - \frac{\rho h\omega^2}{2}\int_{A_p} W'^2\,\mathrm{d}x'\,\mathrm{d}y', \tag{3a}$$

W' is the true displacement amplitude of the plate; the first integral in equation (3a), taken over the actual area of the plate surface A_p, is the (maximum) strain energy of the plate and the second integral measures the maximum kinetic energy of the plate.

The functional for the concentrated mass–spring has the form [3]

$$J_m = \frac{k_m}{2}v'^2 - \frac{m\omega^2}{2}(v' + W'_m)^2, \tag{3b}$$

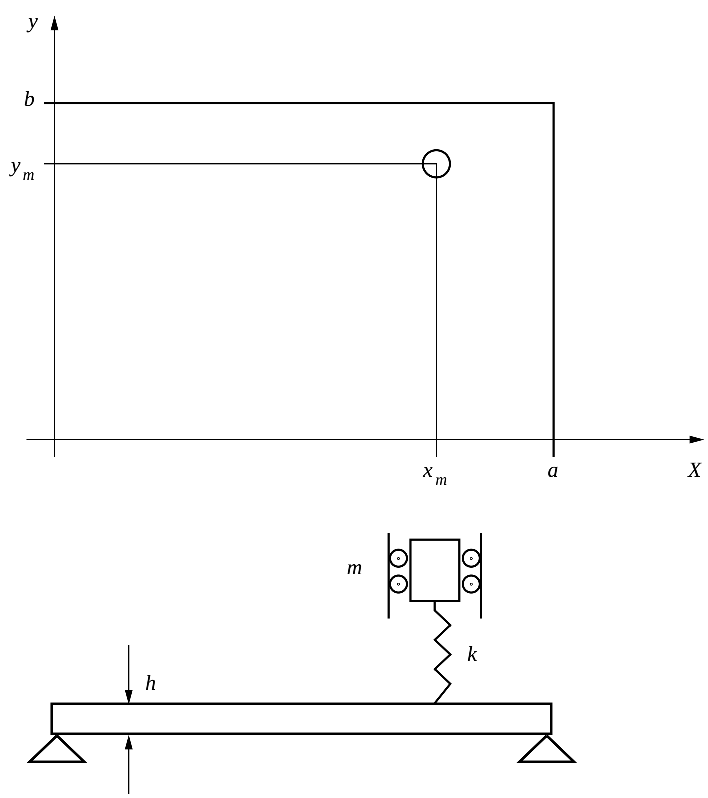

Figure 1. Vibrating system under consideration.

TABLE 1

Values of the first four frequency coefficients Ω_1 to Ω_4 in the case of a bare anisotropic rectangular plate as the number of terms of the approximated displacement function is increased. Five different values of the aspect ratio b/a have been depicted

Aspect ratio	Number of terms	Ω_1	Ω_2	Ω_3	Ω_4
1/2	100	39·62	69·11	113·0	124·0
	225	39·45	68·74	112·5	123·8
	400	39·36	68·57	112·3	123·6
	625	39·30	68·45	112·1	123·6
	900	39·26	68·38	112·0	123·5
2/3	100	27·45	54·67	77·42	91·92
	225	27·32	54·42	77·17	91·59
	400	27·24	54·33	77·03	91·41
	625	27·19	54·25	76·95	91·30
	900	27·16	54·21	76·89	91·23
1	100	18·37	37·07	50·79	61·48
	225	18·28	36·92	50·63	61·28
	400	18·24	36·86	50·54	61·15
	625	18·21	36·81	50·48	61·09
	900	18·18	36·78	50·44	61·04
3/2	100	13·93	23·60	37·23	43·87
	225	13·89	23·50	37·07	43·79
	400	13·86	23·44	37·00	43·76
	625	13·84	23·40	36·95	43·73
	900	13·83	23·38	36·92	43·72
2	100	12·24	18·17	26·58	37·10
	225	12·21	18·10	26·46	36·92
	400	12·19	18·05	26·40	36·85
	625	12·18	18·03	26·36	36·80
	900	12·18	18·01	26·33	36·77

where v' is the mass displacement amplitude relative to the plate; $(v' + W'_m)$ is the total displacement amplitude of the point pass; and W'_m is the displacement amplitude of the plate at the concentrated mass position.

In equations (3a) above D_{ij} are the well known flexural rigidities of the (anisotropic) plate, which, for an isotropic plate, takes the simple form

$$D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^2)}; \quad D_{12} = \nu D_{11}; \quad D_{66} = \frac{(1-\nu)}{2} D_{11}; \quad D_{16} = D_{26} = 0. \tag{4}$$

If the length of the sides of the rectangular plate are a and b in the x and y directions, respectively, equations (3a) and (3b) can be cast in a non-dimensional form by introducing

$$W = W'/a, \quad x = x'/a, \quad y = y'/b, \quad v = v'/a. \tag{5}$$

One gets, for the functional for the whole system of Figure 1,

$$J_{nd} = \frac{2J}{D_{11}} = \int_{A_p} \left[\left(\frac{\partial^2 W}{\partial x^2} \right)^2 + \frac{2}{\eta^2} \frac{D_{12}}{D_{11}} \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + \frac{1}{\eta^4} \frac{D_{22}}{D_{11}} \left(\frac{\partial^2 W}{\partial y^2} \right)^2 \right.$$

$$+ \frac{4}{\eta} \left(\frac{D_{16}}{D_{11}} \frac{\partial^2 W}{\partial x^2} + \frac{1}{\eta^2} \frac{D_{26}}{D_{11}} \frac{\partial^2 W}{\partial y^2} \right) \left(\frac{\partial^2 W}{\partial x \, \partial y} \right)$$

$$\left. + \frac{4}{\eta^2} \frac{D_{66}}{D_{11}} \left(\frac{\partial^2 W}{\partial x \, \partial y} \right)^2 \right] \eta \, \mathrm{d}x \, \mathrm{d}y$$

$$- \Omega^2 \int_{A_p} W^2 \eta \mathrm{d}x \, \mathrm{d}y + K_m v^2 - M \Omega^2 \eta (v + W_m)^2, \tag{6}$$

TABLE 2

Values of the first four frequency coefficients Ω_1 to Ω_4 in the case of an anisotropic rectangular plate with $b/a = 2/3$ for different positions and values of the mass–spring system (Figure 2)

Mass co-ordinates	m/M_p	Ka^2/D_{11}	Ω_1	Ω_2	Ω_3	Ω_4
$x/a = 0{\cdot}50$		1	3·853	27·36	54·33	77·03
$y/b = 0{\cdot}50$	0·1	10	11·58	28·59	54·33	77·03
		∞	22·64	54·33	77·03	79·11
(a)						
		1	2·224	27·36	54·33	77·03
	0·3	10	6·729	28·42	54·33	77·03
		∞	17·50	54·33	68·61	77·03
$x/a = 0{\cdot}75$		1	3·859	27·29	54·38	77·03
$y/b = 0{\cdot}50$	0·1	10	11·78	27·86	54·86	77·07
		∞	24·55	46·62	75·91	87·98
(b)						
		1	2·228	27·29	54·38	77·03
	0·3	10	6·828	27·78	54·84	77·07
		∞	20·06	40·90	74·74	84·89
$x/a = 0{\cdot}50$		1	3·860	27·30	54·34	77·06
$y/b = 0{\cdot}75$	0·1	10	11·81	27·89	54·48	77·34
		∞	24·57	50·80	67·14	91·02
(c)						
		1	2·229	27·30	54·34	77·06
	0·3	10	6·843	27·80	54·48	77·33
		∞	20·28	45·36	61·98	90·59
$x/a = 0{\cdot}75$		1	3·862	27·28	54·34	77·07
$y/b = 0{\cdot}75$	0·1	10	11·89	27·69	54·42	77·46
		∞	25·30	51·42	63·19	89·90
(d)						
		1	2·229	27·28	54·34	77·07
	0·3	10	6·883	27·63	54·41	77·45
		∞	21·61	44·08	58·13	89·19

TABLE 3

Values of the first four frequency coefficients Ω_1 to Ω_4 in the case of an anisotropic square plate for different positions and values of the mass–spring system (*Figure* 2)

Mass co-ordinates	m/M_p	Ka^2/D_{11}	Ω_1	Ω_2	Ω_3	Ω_4
$x/a = 0.50$		1	3·138	18·35	36·86	50·54
$y/b = 0.50$	0·1	10	9·178	19·66	36·86	50·54
		∞	15·20	36·86	50·54	53·25
(a)						
		1	1·812	18·35	36·86	50·54
	0·3	10	5·368	19·41	36·86	50·54
		∞	11·80	36·86	46·47	50·54
$x/a = 0.75$		1	3·147	18·29	36·87	50·56
$y/b = 0.50$	0·1	10	9·457	18·93	37·01	50·82
		∞	16·48	34·38	44·92	60·74
(b)						
		1	1·816	18·29	36·87	50·56
	0·3	10	5·501	18·80	37·01	50·81
		∞	13·66	30·69	41·96	60·28
$x/a = 0.50$		1	3·145	18·29	36·91	50·54
$y/b = 0.75$	0·1	10	9·413	18·92	37·38	50·59
		∞	16·41	31·67	49·65	58·91
(c)						
		1	1·816	18·29	36·91	50·54
	0·3	10	5·476	18·79	37·36	50·58
		∞	13·43	27·82	48·95	57·12
$x/a = 0.75$		1	3·149	18·27	36·86	50·58
$y/b = 0.75$	0·1	10	9·558	18·71	36·94	50·97
		∞	16·97	34·89	41·94	60·08
(d)						
		1	1·818	18·27	36·86	50·58
	0·3	10	5·548	18·62	36·93	50·96
		∞	14·57	29·79	39·05	59·55

where as usual, $\Omega^2 = \rho h \omega^2 a^4 / D_{11}$ is the non-dimensional frequency coefficient; $M = m/M_p$, M_p being the total mass of the plate; $K_m = k_m a^2/D_{11}$ is the non-dimensional mass–spring constant and $\eta = b/a$ is the aspect ratio of the plate.

Expressing the displacement amplitude $W(x, y)$ of the plate in an approximate way by means of a double Fourier series [4],

$$W_a(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{mn} \sin(m\pi x) \sin(n\pi y), \tag{7}$$

one gets as a final expression for the functional the sum

$$\begin{aligned} J_{nd} = \eta \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{l=1}^{N} \sum_{k=1}^{M} b_{kl} b_{mn} \Bigg\{ \pi^4 \left[m^2 k^2 + 2 \frac{D_{12}}{D_{11}} \frac{k^2 n^2}{\eta^2} + \frac{D_{22}}{D_{11}} \frac{n^2 l^2}{\eta^4} \right] A_{SS} \\ - 4\pi^4 \left[\frac{k^2 mn}{\eta} \frac{D_{16}}{D_{11}} + \frac{l^2 mn}{\eta^3} \frac{D_{26}}{D_{11}} \right] A_{SC} + \pi^4 \left[\frac{4 D_{66}}{D_{11}} \frac{klmn}{\eta^2} \right] A_{CC} - \Omega^2 A_{SS} \Bigg\} \\ + K_m v^2 - M \Omega^2 \eta (v + W_m)^2, \end{aligned} \tag{8}$$

where

$$W_m = W_a(x_m, y_m) = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{mn} \sin(m\pi x_m) \sin(n\pi y_m). \tag{9}$$

is the amplitude displacement of the plate at the mass position, x_m and y_m being the non-dimensional mass position co-ordinates. Also

$$A_{SS} = \int_{A_p} \sin(k\pi x) \sin(l\pi y) \sin(m\pi x) \sin(n\pi y) \, \mathrm{d}x \, \mathrm{d}y, \tag{10}$$

$$A_{SC} = \int_{A_p} \sin(k\pi x) \cos(m\pi x) \sin(l\pi y) \cos(n\pi y) \, \mathrm{d}x \, \mathrm{d}y, \tag{11}$$

$$A_{CC} = \int_{A_p} \cos(k\pi x) \cos(l\pi y) \cos(m\pi x) \cos(n\pi y) \, \mathrm{d}x \, \mathrm{d}y. \tag{12}$$

TABLE 4

Values of the first four frequency coefficients Ω_1 to Ω_4 in the case of an anisotropic rectangular plate with $b/a = 3/2$ for different positions and values of the mass–spring system (*Figure* 2)

Mass co-ordinates	m/M_p	Ka^2/D_{11}	Ω_1	Ω_2	Ω_3	Ω_4
$x/a = 0{\cdot}50$		1	2·558	13·96	23·44	37·03
$y/b = 0{\cdot}50$	0·1	10	7·351	15·12	23·44	37·36
		∞	11·52	23·44	31·99	43·76
(a)						
		1	1·477	13·96	23·44	37·03
	0·3	10	4·316	14·87	23·44	37·35
		∞	8·878	23·44	28·69	43·76
$x/a = 0{\cdot}75$		1	2·567	13·91	23·44	37·01
$y/b = 0{\cdot}50$	0·1	10	7·635	14·50	23·48	37·12
		∞	12·51	23·07	32·21	39·61
(b)						
		1	1·482	13·91	23·44	37·01
	0·3	10	4·453	14·36	23·48	37·11
		∞	10·36	22·25	27·33	38·89
$x/a = 0{\cdot}50$		1	2·563	13·91	23·50	37·02
$y/b = 0{\cdot}75$	0·1	10	7·520	14·45	24·08	37·19
		∞	12·36	20·17	34·78	43·71
(c)						
		1	1·480	13·91	23·50	37·02
	0·3	10	4·389	14·33	24·03	37·19
		∞	9·834	18·26	33·62	43·68
$x/a = 0{\cdot}75$		1	2·569	13·89	23·46	37·00
$y/b = 0{\cdot}75$	0·1	10	7·708	14·27	23·69	37·00
		∞	12·90	21·58	35·65	37·33
(d)						
		1	1·483	13·89	23·46	37·00
	0·3	10	4·483	14·18	23·67	37·00
		∞	10·96	19·35	31·21	37·10

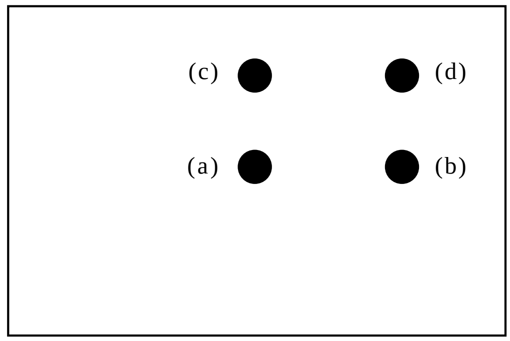

Figure 2. Positions of the mass–spring system on the vibrating anisotropic simply supported plate.

In order to minimize the functional one has to take its partial derivatives with respect to the coefficients b_{ij} and v in expression (8) and equate these derivatives to zero. That is to say,

$$\frac{\partial J}{\partial b_{11}} = 0, \quad \frac{\partial J}{\partial b_{12}} = 0, \ldots, \quad \frac{\partial J}{\partial b_{MN}} = 0, \quad \frac{\partial J}{\partial v} = 0.$$

This yields an $(M \times N + 1)$ homogeneous linear system of equations in the b_{ij}'s and the v. A secular determinant in the natural frequency coefficients of the system results from the non-triviality condition.

The present study is concerned with the determination of the first four frequency coefficients, Ω_1 to Ω_4, in the case of anisotropic rectangular plates carrying elastically mounted concentrated masses.

3. NUMERICAL RESULTS

Tables 1 to 4 depict calculations performed for anisotropic simply supported rectangular plates of uniform thickness, with varying aspect ratios, and for several different positions of the mass–spring system, taking $D_{12}/D_{11} = 0{\cdot}3$, $D_{22}/D_{11} = 0{\cdot}5 = D_{66}/D_{11}$, and $D_{16}/D_{11} = 1/3 = D_{26}/D_{11}$. Table 1 shows the first four frequency coefficients for a bare anisotropic rectangular plate with five different values of its aspect ratio. As the depicted values show, the convergence in the Ω_i's is quite satisfactory as the number of terms in the displacement function is increased from 100 (i.e., $M = 10 = N$) to 900 terms. As usual, special care has been taken to manipulate such large determinants and 80 bit floating point variables (IEEE-standard temporary reals) have been used in order to obtain accurate results.

Tables 2–4, on the other hand, show the same frequency coefficients for the case of an anisotropic simply supported rectangular plate of uniform thickness when coupled to a mass–spring system. Four different and representative positions of the mass–spring system have been taken into account.

In this last case, $M = 20 = N$ have been used in the Fourier series approach, that is to say a secular determinant of order 401 was posed for all the situations at hand.

Table 2 shows fundamental frequency coefficients in the case of a square plate while Tables 1 and 3 illustrate the case of rectangular plates with aspect ratios $b/a = 2/3$ and $3/2$, respectively.

The present approach can be extended in a straightforward fashion to the case of plates of non-uniform thickness, presence of orifices, etc. In the case of other kinds of boundary conditions, one would use the corresponding combinations of "beam functions" popularly used when dealing with isotropic and orthotropic structural elements [5].

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